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REACTIONS OF DERMAL MELANOPHORES IN NECTURUS TO HEAT AND TO COLD

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The experiments discussed in this paper deal with changes seen in the dermal melanophores of Necturus under temperature variation. In the work of previous investigators there has been some uncertainty on this point, due, no doubt, to the fact that in Necturus the melanophores do not react to heat and cold in the usual way. Thus in most amphibians the melanophores are known to respond to heat by contraction and to cold by expansion. Dawson (1920, p. 523) stated that the dermal melanophores of Necturus reach their maximum contraction at temperatures from 9° to 10°C., their maximum expansion at 25°, and showed that between these two extremes a progressive change occurs toward full expansion in rising, toward complete contraction in falling temperature, exactly the reverse of the responses commonly observed in other amphibians. Hogben (1927, p. 49) called attention to this unusual behavior, as observed by Dawson. It seemed, therefore, worth while to reinvestigate this question. The problem was suggested to me by Professor G. H. Parker, under whose direction my work has been carried on.

Experimentally, adult Necturus is unfavorable, since each individual has its fixed color pattern in which the reactions of the melanophores, if any, are too obscured for precise observation. For excellent material I am indebted to Miss Madeleine Parker Grant, who allowed me the use of young specimens of Necturus, approximately ten months old. In these the dermal melanophores of the dorsal trunk are confined to three, discrete, longitudinal stripes separated by bright-yellow bands of xanophores with a sprinkling of epidermal, black pigment-cells in tiny compact groups. The stripes of black are never packed so densely as to obscure observation on individual melanophores, even when the reactions are slight.

Animals were placed in dishes of water of appropriate temperatures and their subsequent responses observed under the microscope. Jars,

one wrapped in white paper, another in black, under equal overhead illumination gave uniform light and dark backgrounds. As a dark-chamber a small, covered jar having inlet and outlet for water and made light-proof served satisfactorily in those cases in which temperature had to be controlled. By means of connecting tubes and stopcocks a steady flow of water could be maintained at any desired temperature. This could be controlled both by the water in the supply tank and in the exhaust, the jar remaining unopened during the experiment. Best results were obtained when animals were left undisturbed in their experimental dishes for two hours. In this paper temperatures from 1° to 30°C. are considered. Three main stages in melanophore change—(1) full expansion, (2) moderate contraction, and (3) complete contraction—as shown in

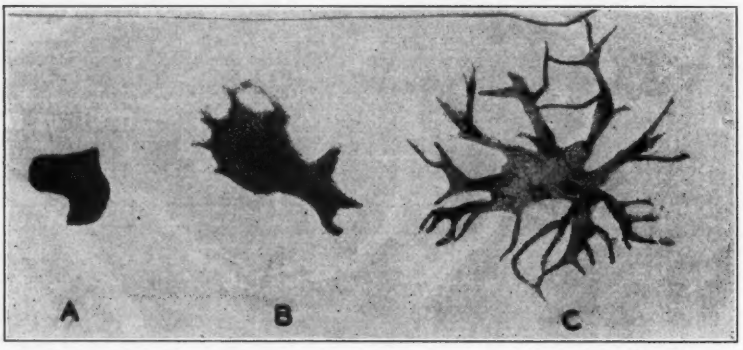

FIGURE 1
Dermal melanophores of Necturus. A, fully contracted; B, moderately expanded; C, fully expanded.

figure 1 will serve as standards of comparison. Such a degree of contraction as is often seen in fishes in which the melanophore resembles a minute dot was never found in my experiments with Necturus and probably never occurs in this animal.

At equal and full illumination, background has no observable effect on the color pattern of either adult or young Necturus. If all illumination is cut off for three hours the adults will appear somewhat lighter, especially around the jaws, gills, and forelimbs. But these changes in the adult are so difficult to follow that for all detailed experiments only young animals nine to ten months of age were used.

That strong direct light is a more effective stimulus than background can be proved by the fact that only under mild illumination the melanophore configuration appears less dense on a white background than on a black one. Toward evening the bodies of the young animals look bleached.

By keeping Necturus for two hours in a dark-chamber, this condition of moderate contraction (Fig. 1B) predominates most strikingly around the gills, laterally along the jaws and forelimbs, far less at the posterior end. Since Dawson (1920, p. 521) found no secondary responses, I was astonished to note that after twelve hours in a completely dark chamber the expanded state of the melanophore reappears.

Temperature effects, like those produced by the background, are observed over a range from 2° to 30°C. by the reaction to strong illumination, namely, full expansion. Therefore those experiments carried out in the continuous-flow dark-chamber are alone of importance. To quote from Dawson (1920, p. 523): "high temperatures, similarly to light, cause an expansion of the melanophores," and "....in darkness at low temperatures (9° to 10°C.) maximum contraction of melanophores always results." When I tested low temperatures—2°C. and a little higher—I found that these temperatures bring about full expansion instead of the expected contraction, reversing the response to darkness. Between 6° and 9°C. this expanded state passes over into one of moderate contraction, the posterior portion of the head changing far more gradually than other parts of the animal. At the high temperatures, 28° to 30°C., full expansion is usual and at 26° to 23°C. varying degrees of moderate contraction. The state of moderate contraction produced by intermediate temperatures is changed at extremely high and extremely low temperatures for one of full expansion.

Whereas absence of light at room temperature permits of no more than a moderate contraction as in figure 1B, intracoelomic injection of adrenalin brings the melanophores to complete contraction as seen in A. Three-year old larvae injected with 0.2 cc. of 10 per cent commercial adrenalin showed after two and a half hours complete contraction of the melanophores on their ventral surfaces. Injection of obstetrical pituitrin—0.2 cc.—brought about in the same time full expansion of the melanophores. One hour after implanting fresh frog's anterior lobe and pars intermedia into nine-month old Necturus, Miss Grant observed that despite the dark-chamber and room temperature the melanophores were completely expanded.

There is no conclusive evidence for or against the belief that the reactions of the melanophore in Necturus are under the control of the nervous system. Probably they react directly to light, temperature, hormones, and other stimuli. Since there is a reversal of the response to darkness after twelve hours, from mild contraction to full expansion, reaction to background may possibly be under nerve control and require very much more time to become effective. As far as I have tested them, melanophores of denervated pieces of skin respond in every way as do those in the whole animal.

Full expansion and complete contraction are, then, the results of such stimuli as light, extremes of heat and cold, pituitrin, and adrenalin. Moderate contraction (B) might well be called a state of relaxation when the dermal melanophores are under no stimulation—room temperature and darkness being taken as the absence of all stimuli.

My results on Necturus confirm those of Dawson in that the melanophores of this animal are expanded when the temperature is high (28° to 30°C.) and moderately contracted when it is intermediate or low (26° to 6°C.). They are novel in that they show expansion at 2°C., a temperature lower than any tried by Dawson. Necturus, then, agrees with other amphibians in that its melanophores expand at very low temperatures and contract at moderately high ones. It is unlike other amphibians, however, in that these cells expand when subjected to relatively high temperatures.

Dawson, A. B., "The Integument of Necturus Maculosus," *J. Morph.*, **34**, 487–589 (1920).

Hogben, L. T., "The Comparative Physiology of Internal Secretion," Cambridge, 148 pp. (1927).

HISTOLOGICAL POLARIZATION OF LATERAL GIANT-FIBERS IN THE CRAYFISH

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It is well known that annelids and crustaceans possess giant-fibers in their central nervous systems. Johnson[1] ('24) first demonstrated that the lateral giant-fibers in crustaceans were not continuous, but segmented in character, the interruptions occurring in the ganglia.

This fact was further demonstrated in the earthworm by Stough[2] ('26) who also showed that one segment of a given giant-fiber is distinguished from the other with which it overlaps by a deeper coloration. This was not noted by Johnson in the crayfish. I was led, therefore, to look for this histological polarization in the crayfish as brought out by Stough in annelids.

In preparing my material I used a modification of the Vom Rath fluid similar to that employed by Johnson ('24) who substituted a few drops of a five per cent solution of acetic acid in place of the one per cent solution required by the original formula. The crayfish was operated upon as quickly as possible. The abdominal portion of the nerve-cord was removed, care being taken to cause as little injury as possible to the fibers.

The best results were obtained by allowing the nerve-cord to remain in the Vom Rath fluid between twelve and fifteen hours. The tissue was then washed in water for an hour and a half, transferred to a five per cent solution of pyrogallic acid and allowed to remain there for twenty-four hours. It was then dehydrated, cleared, embedded in paraffin, and cut at 5 microns.

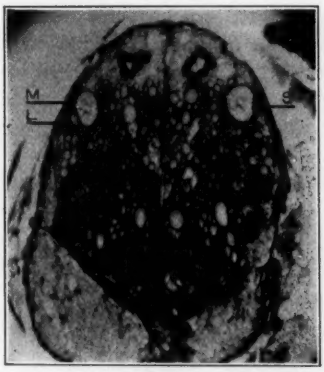

FIGURE 1

Transverse section through a ganglion of the crayfish. L, lateral overlapping segment of the lateral giant-fiber showing dark staining; M, median overlapping segment of the lateral giant-fiber showing light condition; S, septum.

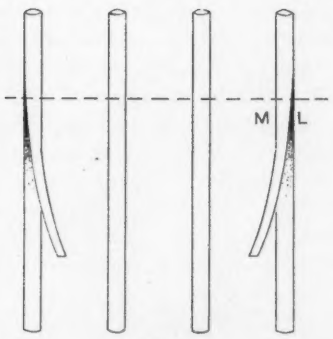

FIGURE 2

Diagrammatic reconstructions of lateral fibers in the region of a ganglion; the dotted line represents the level at which the section for figure 1 was cut; L, lateral segment; M, median segment.

The segmental character of the lateral, overlapping fiber is shown in the ganglion by a color differentiation (Fig. 1, *M*, *L*). In the ganglionic region the lateral member of the overlapping pair is always darker than the median one, and these two members are separated by a distinct septum (Fig. 1, *S*). The relations of these members as seen in a longitudinal reconstruction is shown in figure 2.

Conclusion.—The segments in the lateral giant-fibers of the crayfish show histological polarity as demonstrated by their staining reactions.

[1] Johnson, G. E., "Giant Nerve Fibers in Crustaceans, with Special Reference to Cambarus and Palaemonetes," *J. Comp. Neur.*, **36**, 323–373 (1924).

[2] Stough, H. B., "Giant Nerve Fibers of the Earthworm," *J. Comp. Neur.*, **40**, 409–463 (1926).

ON THE GROUPS OF ORIENTABLE TWO-MANIFOLDS

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1. *Introduction.*—The group of the orientable two-manifold of genus $p > 1$ has been investigated at length by Nielson,[2] but his work rests on an *a priori* construction of the universal covering surface allied with considerations of non-Euclidian geometry. In the present paper, we initiate and present briefly a direct intrinsic investigation of the same question. This method has the evident advantage of all intrinsic methods: it brings out clearly the basic elements of the problem.

2. *Products of Generators of the Group.*—The group of an orientable two-manifold of genus $p > 0$ is generated by $4p$ distinct operations: a_i, b_i, $i = 1, 2, \ldots, p$, and their inverses; with the single *fundamental relation*

$$R \equiv \prod_{i=1}^{p} a_i b_i a_i^{-1} b_i^{-1} = 1.$$

The cyclic orders represented by R and R^{-1} will be designated by 0_1 and 0_2, respectively. The cyclic order represented by $\prod_{i=1}^{p} a_i b_i^{-1} a_i^{-1} b_i$ will be designated by 0_3.

By a product we mean always a product of generators. The identity standing by itself is not a product. A product of any $2p$ consecutive generators of 0_1 or 0_2 will be called *a direct or an inverse semicycle*, respectively. A direct or an inverse semicycle whose first generator is g will be designated by (G) or $(G)'$, respectively.

A product will be said to be *reduced* if it has the following three properties:

A. No generator immediately follows its inverse.

B. There are no more than $2p$ consecutive generators in the order 0_1 or 0_2.

C. If there is a direct [inverse] semicycle $g_1 g_2 \ldots g_{2p}$, the generator immediately following this semicycle in the product is one of the $2p-1$ generators preceding [succeeding] g_{2p}^{-1} in the order 0_3.

Two products are said to be *equal* if one can be derived from the other by means of the relations:

$$g\, g^{-1} = 1, \quad f\, R f^{-1} = 1,$$

where g is a generator and f a finite product. A product is said to be *reducible*, if there exists an equal reduced product; *non-reducible* otherwise. We have necessary and sufficient conditions for reducibility of finite and infinite products, and existence theorems for reducible and

non-reducible finite and infinite products. We prove also that a reduced finite product has either one equal but not identical reduced finite product or else none; and show what type of reduced finite product has an equal one and to what type this equal product belongs. From the considerations of reduced finite products we can easily obtain similar conclusions for reduced infinite products.

For an even p, the reduced infinite products with the periods (G) or $(G)'$ will be called *direct or inverse products*. For an odd p, the reduced infinite products with the periods

$$(A_i),\ (B_i)(B_{i-1})\dots(B_{i-p+1}),\ (A_i^{-1}),\ (B_i^{-1})(B_{i+1}^{-1})\dots(B_{i+p-1}^{-1}),$$

or

$$(A_i)'(A_{i+1})'\dots(A_{i+p-1})',\ (B_i)',\ (A_i^{-1})'(A_{i-1}^{-1})'\dots(A_{i-p+1}^{-1})',\ (B_i^{-1})',$$

will be called *direct or inverse products*.[3] Each direct product has one and only one equal but not identical reduced infinite product and this latter is inverse. Any other reduced infinite product has none. We shall say that two infinite products are *equivalent* when they are identical beyond a certain point. Then obviously there are $4p$ non-equivalent direct [inverse] products for an even p and only $2p + 2$ non-equivalent ones for an odd p.

3. *Representation of Reduced Infinite Products by Points of a Circle.*—Divide a circle into $4p$ non-overlapping closed intervals, and then subdivide each interval into a definite number of non-overlapping closed intervals. By dividing repeatedly the intervals and associating them with reduced finite products according to a rule based on the three properties of reduced product, we can associate a sequence of mutually inclusive closed intervals on the circle with each reduced infinite product. The limit point of the sequence will be taken to represent the product. All reduced infinite products are uniquely represented by points of the circle. We show that a point of the circle, not a point of division, represents a reduced infinite product which is neither direct nor inverse, and that a point of division on the circle represents two equal but not identical reduced infinite products, one of which is direct and the other inverse. Hence we have a *one-to-one correspondence between the points of the circle and the non-identical reduced infinite products.*

The points of division are everywhere dense on the circle. Hence the points which represent reduced infinite products equivalent to $4p[2p + 2]$ direct or inverse products for an even [odd] p are everywhere dense on the circle.

4. *Infinite Net. A Closed Two-Cell.*—With the group we can associate an infinite net N of $4p$-sided polygons such that any vertex of N is incident with $4p$ sides and $4p$ polygons. All products, finite and infinite, are

represented uniquely by paths, which are made up of the sides of the net and whose initial points coincide with the same vertex of N, arbitrarily chosen. The terminal points of the paths representing two equal finite products coincide with the same vertex of N.

We can associate with the net N an infinite two-manifold which can be mapped topologically on the interior E_2 of a circle S. The points of S may be taken to represent the reduced infinite products and thus the infinite paths of the transformed net representing these products, and will be called the *ideal elements of the infinite two-manifold.* By a proper definition of continuity on $E_2 + S$, we can prove *that the infinite two-manifold and its ideal elements defined by means of the group is a closed two-cell.*

[1] Research Fellow of China Foundation. The author wishes to thank Professor S. Lefschetz for valuable suggestions and encouragement in connection with this investigation.

[2] Nielson, J., *Acta Mathematica*, **50**, 189–379 (1927). This is the third of his four papers on this subject.

[3] We agree to set $(G_m) \equiv (G_n)$ and $(G_m)' \equiv (G_n)'$, if $m = cp + n$, c being an integer and $1 \leqq n \leqq p$.

INVERSE COMMUTATOR SUBGROUPS

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If s and t represent any two operators of a given group G then the operator $s^{-1}t^{-1}st$ is commonly called the commutator of s and t. In the present article the operator $s^{-1}t^{-1}st^{-1}$ is defined as the *inverse commutator* of s and t and some fundamental properties of these commutators are developed. Since the transform of an inverse commutator by any operator of the group is an inverse commutator of this group it results directly that all the inverse commutators of a group generate an invariant subgroup of this group which will be called the *inverse commutator subgroup.* The corresponding quotient group cannot involve any operator whose order exceeds 2 and hence it must be the abelian group of order 2^m, and of type $(1, 1, 1, \ldots)$. Since all the inverse commutators of such a group are obviously equal to the identity it results directly that *the inverse commutators of a group generate its smallest invariant subgroup which gives rise to an abelian quotient group of order* 2^m *and of type* $(1, 1, 1, \ldots)$, *and every invariant subgroup which gives rise to such an abelian quotient group must involve the inverse commutator subgroup.*

From this theorem it results directly that the inverse commutator subgroup of every group whose order is of the form 2^m is its ϕ-subgroup and if the index of this subgroup is 2^α then the number of operators in every set of independent generators of the group is exactly α, and vice versa. A necessary and sufficient condition that the commutator of s and t is the identity is that s and t are commutative while a necessary and sufficient condition that the inverse commutator of s and t is the identity is that s transforms t into its inverse. It should be noted that when s transforms t into its inverse it is not necessarily true that t also transforms s into its inverse. The concept of inverse commutator seems to be especially useful when t is supposed to represent successively all the operators of an abelian group H which is transformed according to an automorphism of order 2 by s. Hence we shall assume in what follows that s and t satisfy these conditions so that only special inverse commutators and special inverse commutator subgroups will be under consideration unless the contrary is stated.

Since H is abelian all of its operators which correspond to their inverses under an automorphism of H must constitute a subgroup of H. The inverse commutator subgroup which corresponds to an automorphism of H is therefore simply isomorphic with the quotient group of H with respect to the subgroup formed by all the operators of H which correspond to their inverses under this automorphism. In particular, when this automorphism is of order 2 the corresponding inverse commutator subgroup must be composed of operators which correspond to themselves under this automorphism. It therefore results that when H is of odd order its operators which correspond to their inverses under an automorphism of order 2 constitute a subgroup which has only the identity in common with the inverse commutator subgroup resulting from this automorphism. The concept of inverse commutator subgroup therefore furnishes a direct proof of the theorem that every abelian group of odd order is the direct product of the two subgroups composed respectively of its operators which correspond to themselves and to their inverses under an automorphism of order 2.[1]

If H is a cyclic group of order p^m, p being a prime number, it results directly from the preceding paragraph that its group of isomorphisms involves only one operator of order 2 when $p > 2$. When $p = 2$ and $m > 2$ it results from similar considerations that this group of isomorphisms involves three and only three operators of order 2. In fact, if the order of the inverse commutator subgroup in an automorphism of order 2 of this group exceeds 2 it must be 2^{m-1} since the operators of order 4 must then correspond to themselves. As all the operators of this commutator subgroup must also correspond to themselves under this automorphism it results that either all the operators correspond to themselves or that

only the operators of highest order are multiplied by the operator of order 2 when the inverse commutator subgroup corresponding to an automorphism of order 2 of this group is of order 2^{m-1}. There is obviously one such automorphism of order 2 when the inverse commutator subgroup is the identity and one when this subgroup is of order 2.

As a further illustration of the use of these special inverse commutator subgroups, and on account of the results, we shall employ them to determine all the groups which have the property that they involve a given abelian subgroup H of index 2 while all the additional operators are of order 4. Since $stst = s^2s^{-1}tst = s^2t_0$, where t_0 is an inverse commutator of G, and since t_0 is commutative with s, it results that t_0 is either the identity or of order 2. It cannot be equal to s^2 since st is assumed to be of order 4. This proves the following theorem: *If a group contains an abelian subgroup H of index 2 and all of its remaining operators are of order 4 then its inverse commutators constitute a subgroup involving no square of an operator not found in H. If this subgroup is extended by such a square there results a subgroup of the central which involves no operator whose order exceeds 2 and only such squares in addition to the inverse commutator subgroup.*

On the other hand it is easy to prove that when an abelian group H contains a subgroup K involving no operator whose order exceeds 2 and K is contained in a subgroup of H corresponding to a quotient group which is isomorphic to a subgroup of index 2 in K, then H can be extended to a group G of twice the order of H and such that each of the additional operators is of order 4. The number of the different squares of these additional operators is always equal to the order of the inverse commutator subgroup of H under G. In particular, a necessary and sufficient condition that all these additional operators have the same square is that s transforms every operator of H into its inverse. If such a G can be constructed so that it has 2^α, $\alpha > 0$, such distinct squares then that it is always possible to construct with the same H a G which involves any lower power of 2 such distinct squares. The theorem noted at the close of the preceding paragraph involves therefore a necessary and sufficient condition that an abelian group may be extended to a group of twice its order and such that each of the added operators is of order 4.

A necessary and sufficient condition that the inverse commutator subgroup corresponding to an automorphism of G is always identical with the commutator subgroup corresponding to the same automorphism is that G is abelian, of order 2^m, and of type (1, 1, 1, ...). Hence it results directly that the commutators corresponding to an automorphism of order 2 of such a group must correspond to themselves under this automorphism. The special inverse commutator subgroup in which t is an operator of an abelian group H which is transformed into itself by s and

involves s^2 is composed of the smallest subgroup of G which gives rise to a quotient group which is either dihedral or generalized dihedral. Hence it follows that the concepts of inverse commutator subgroup and special inverse commutator subgroup enable us to unify a number of fundamental theorems of groups of finite order. This unification is the main object of the present note. While the identity automorphism gives rise only to the identity commutator it gives rise to the squares of all the operators of the group as inverse commutators.

[1] G. A. Miller, *Trans. Am. Math. Soc.*, **10**, 472 (1909).

THE CONVENTION OF EQUIDIMENSIONAL ELECTRIC AND MAGNETIC UNITS

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It is the object of this paper to present a set of electromagnetic unit dimensions, common to both electric and magnetic systems, based on the conventional hypothesis that the dimensions of permittivity κ_0 and permeability μ_0 for free space are the same.

It was announced by Maxwell in 1881 in his Treatise,[1] Vol. 2, Chap. X, that "every electromagnetic quantity may be defined with reference to the fundamental units of Length, Mass, and Time." He formulated the "dimensions" of each quantity, such as Resistance, Current, Capacitance, etc., in terms of the fundamental quantities of dynamics—L, M, and T. Thus with any velocity V, defined as a ratio L/T, the dimensions of velocity would be L^1, M^0, T^{-1}, or giving only the exponents, as (1, 0, -1). Similarly, the ordinary dynamic formulas of energy W, and of power P, being respectively $MV^2/2$ and W/T, their exponential dimensional formulas would be (2, 1, -2) and (2, 1, -3).

Maxwell also showed that there were always two different dimensional formulas for each electromagnetic quantity: namely, one in the electric (electrostatic) system, and one in the magnetic (electromagnetic) system. The electric units were derived from the force of repulsion between like electric charges across a known distance, assuming that the permittivity κ_0 of free space is the numeric unity. The magnetic units were, however, derived from the force of repulsion between like magnetic poles across a known distance, assuming that the permeability μ_0 of free space is the numeric unity.

The accompanying table 1 is based on Maxwell's list of dimensions for

the electric and magnetic units. Column II names the quantities considered, III gives the customary distinguishing symbol, IV, V and VI the dimensions of the quantity in the magnetic system; while VIII, IX and X give them in the electrostatic system. To take a simple example, No. 9 in the list, Resistance R, has dimensions L^1, M^0, T^{-1}, or those of velocity $V = L/T$ in the magnetic system; but has L^{-1}, M^0, T^1, or those of inverse velocity $V^{-1} = T/L$, or slowness, in the electric system.

Maxwell's findings in these matters, as repeated in Columns IV, V, VI, and VIII, IX, X, have never been disputed, taking into account the postulates he assumed. On the other hand, they have never been regarded as self-explanatory, except perhaps, Electric Capacitance, No. 6, and Magnetic Inductance, No. 18, each of which appears with the dimensions of a simple Length, and for each of which a physical realization has been suggested by several writers.

It was pointed out by Rücker[2] in 1889, that Maxwell's list of unit dimensions was necessarily incomplete; because the constants of free space (permittivity κ_0 and permeability μ_0) are involved in the action of the observed forces between electric charges or magnetic poles. These space constants are suppressed in Maxwell's lists, and yet they must be assumed to have dimensions of some sort. Indeed, Maxwell showed in his Treatise,[1] Chap. XX, that

$$\frac{1}{\sqrt{\kappa_0 \mu_0}} = v, \tag{1}$$

where v is the velocity of electromagnetic propagation, or approximately 300,000 km. per second. Rücker introduced these unresolved space constants, with their appropriate exponents, into Maxwell's dimensional formulas, as indicated in Columns VII and XI of Table 1. Thus Resistance R (No. 9) has revised Maxwell-Rücker dimensions of $D_\mu = L^1$, M^0, T^{-1}, μ^1 in the magnetic system and $D_\kappa = L^{-1}$, M^0, T^1, κ^{-1}, in the electric system. These Maxwell-Rücker dimensions, as repeated in Table 1, under D_μ and D_κ, have been recognized since 1889, and are quoted in various text-books and handbooks.[3,4,5,6].

From Maxwell's equation (1), we have:

$$\kappa_0 \mu_0 = \frac{1}{v^2} \tag{2}$$

or the product of the two space constants is the inverse square of the propagation velocity. This equation admits, of course, of an infinite number of physical and dimensional solutions for κ_0 and μ_0, separately. In the electric system of units, κ_0 is taken as unity and μ_0 as $1/v^2$; whereas in the magnetic system, μ_0 is taken as unity and κ_0 as $1/v^2$.

TABLE 1

DIMENSIONAL FORMULAS OF ELECTROSTATIC, ELECTROKINETIC AND MAGNETIC QUANTITIES

I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV
			DIMENSIONAL FORMULA								RATIO D_μ/D_κ		
			D_μ				D_κ						
NO.	QUANTITY	SYMBOL	L	M	T	μ_0	L	M	T	κ_0	L	T	$(\mu_0\kappa_0)$
1	Electromotive force	E	3/2	1/2	−2	1/2	1/2	1/2	−1	−1/2	1	−1	1/2
2	Electric intensity or Electric force	H_e	1/2	1/2	−2	1/2	−1/2	1/2	−1	−1/2	1	−1	1/2
3	Electric quantity or flux	Q	1/2	1/2	0	−1/2	3/2	1/2	−1	1/2	−1	1	−1/2
4	Surface- or flux-density		−3/2	1/2	0	−1/2	−1/2	1/2	−1	1/2	−1	1	−1/2
5	Elastance	S	1	0	−2	1	−1	0	0	−1	2	−2	1
6	Permittance or capacitance	C	−1	0	2	−1	1	0	0	1	−2	2	−1
7	Elastivity		2	0	−2	1	0	0	0	−1	2	−2	1
8	Permittivity	κ	−2	0	2	−1	0	0	0	1	−2	2	−1
9	Resistance	R	1	0	−1	1	−1	0	1	−1	2	−2	1
10	Conductance	G	−1	0	1	−1	1	0	−1	1	−2	2	−1
11	Resistivity	ρ	2	0	−1	1	0	0	1	−1	2	−2	1
12	Conductivity	γ	−2	0	1	−1	0	0	−1	1	−2	2	−1
13	Current density		−3/2	1/2	−1	−1/2	−1/2	1/2	−2	1/2	−1	1	−1/2
14	Current / Magnetomotive force	I / F	1/2	1/2	−1	−1/2	3/2	1/2	−2	1/2	−1	1	−1/2
15	Magnetising force	H	−1/2	1/2	−1	−1/2	1/2	1/2	−2	1/2	−1	1	−1/2
16	Pole strength or flux	m / Φ	3/2	1/2	−1	1/2	1/2	1/2	0	−1/2	1	−1	1/2
17	Surface or flux density	B	−1/2	1/2	−1	1/2	−3/2	1/2	0	−1/2	1	−1	1/2
18	Inductance / Permeance	L / P	1	0	0	1	−1	0	2	−1	2	−2	1
19	Reluctance	R	−1	0	0	−1	1	0	−2	1	−2	2	−1
20	Reluctivity	ν	0	0	0	−1	2	0	−2	1	−2	2	−1
21	Permeability	μ	0	0	0	1	−2	0	2	−1	2	−2	1

Electrostatic: nos. 1–8; Electrokinetic: nos. 9–13; Magnetic: nos. 14–21

Although several different suggestions have been made at different times, for the solution of Maxwell's equation (1), based on physical possibilities; yet, setting physical considerations aside, the simplest solution is evidently $\kappa_0 = \mu_0 = 1/v$, or L^{-1}, M^0, T^1. That suggestion has been made;[7] but so far as the writer is aware, without pointing out the results of this convention upon the series of electric and magnetic unit dimensions, such as those in table 1. If we divide D_μ by D_κ, as in Columns XII, XIII and XIV, we obtain a series which shows that not only are these ratios all dimensionless, since $(\mu\kappa) = L^{-2}T^2$; but also that on the equidimensional convention, $D_\mu = D_\kappa$, or the two series D_μ and D_κ of electric and magnetic dimensions, each of four elements (L, M, T and κ_0 or L, M, T and μ_0) coalesce into a single three-element series (L, M, T) of the Maxwellian type. Thus the equidimensional convention for κ_0 and μ_0 leads to an equidimensional series for all the existing units. Moreover, it seems that the equidimensional solution of the Maxwell equation (1) is the only one which yields equidimensional electric and magnetic series in table 1.

Table 2 shows, in somewhat rearranged form, the single series of dimensions for both electric and magnetic units belonging to any of the recognized systems (C. G. S., and Practical) as obtained from table 1, when μ_0 and κ_0 are replaced by $L^{-1}T^1$, the dimensions for the propagation slowness of approximately $1/(3 \times 10^8)$ seconds per meter. Columns I and XV contain the successive entry numbers, with their dimensions in VII, VIII, and IX. Columns II and XIV give characteristic or descriptive dimensions. Columns III and V give the electrokinetic and electrostatic quantities defined. Their ordinary symbols appear in IV and VI. Columns XI and XIII likewise give the magnetostatic and magnetokinetic quantities. The former pertain to the ordinary steady magnetic circuit, and the latter to the alternating magnetic circuit.

(1) El. Energy EQ, and Mag. Energy FΦ have (2, 1, -2), in common with dynamic energy W.

(2) El. Power E^2G, EI, or I^2R, and Mag. Power F $\dfrac{d\Phi}{dt}$, have (2, 1, -3), in common with dynamic power $P = W/T$.

(3) El. Permittivity κ and Mag. Permeability μ have, by assumption, the slowness dimensions (-1, 0, 1).

(4) Elastivity $1/\kappa$ and Reluctivity $\nu = 1/\mu$ have, by the same assumption, the velocity dimensions (1, 0, -1).

(5) Both Electric and Magnetic Motive Forces, E, F, and potentials V, as well as their currents I, $\dfrac{dQ}{dt}$ and $\dfrac{d\Phi}{dt}$, have (1, 1/2, $-3/2$), the dimensions of root power $\sqrt{P}$.

(6) Elastance S and Reluctance R have (0, 0, -1), of inverse time.

TABLE 2

DIMENSIONS OF ELECTRIC AND MAGNETIC QUANTITIES (STATIC AND KINETIC) ON THE CONVENTION OF EQUIDIMENSIONAL κ_0 AND μ_0

I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV	XV
		ELECTRIC				EQUIDIMENSIONAL FORMULA				MAGNETIC				
NO.	CHARACTERISTIC FORMULA	KINETIC		STATIC		L	M	T	STATIC		KINETIC		CHARACTERISTIC FORMULA	NO.
1	W	Energy	W	Energy $EQ = W$		2	1	−2	$W = F\Phi$ Energy		$W = F\Phi$ Energy		W	1
2	$P = \frac{W}{T}$	Power $\left\{\begin{matrix} E^2G \\ EI \\ I^2R \end{matrix}\right\} = P$		Power $E\frac{dQ}{dt} = P$		2	1	−3	$P = F\frac{d\Phi}{dt}$ Power		$P = F\frac{d\Phi}{dt}$ Power		$P = \frac{W}{T}$	2
3	$1/v$	.....		Permittivity	κ	−1	0	1	μ	Permeability	.....		$1/v$	3
4	v	.....		Elastivity	$1/\kappa$	1	0	−1	ν	Reluctivity	.....		v	4
5	$\sqrt{P}$	Electromotive force	E	E.M.F.	E	1	1/2	−3/2	F	M.M.F.	F	M.M.F.	$\sqrt{P}$	5
	$\sqrt{P}$	Potential	V	Potential	V					Potential		Mag. potential	$\sqrt{P}$	
	$\sqrt{P}$	Current	I	Current $\frac{dQ}{dt} = I$					$\frac{d\Phi}{dt}$	Mag. current	$\frac{d\Phi}{dt}$	Mag. current	$\sqrt{P}$	
6	$1/T$	.....		Elastance	S	0	0	−1	R	Reluctance	.....		$1/T$	6
7	T	El. time const.	CR	Capacitance	C	0	0	1	P	Permeance	L	Inductance	T	7
											L/R	Mag. time constant	T	
8	$\sqrt{T.W}$ or $T\sqrt{P}$	Quantity	IT	Charge	Q	1	1/2	−1/2	m	Pole strength	m	Mag. quantity	$\sqrt{TW}$ or $T\sqrt{P}$	8
		.....		Static flux	Φ_e				Φ	Mag. flux	.....		.....	
9	$\sqrt{TW}/S$ or $T\sqrt{P}/S$	.....		Surf. density	σ	−1	1/2	−1/2	m/S	Pole density	.....		$\sqrt{TW}/S$ or $T\sqrt{P}/S$	9
		.....		Flux density	B_e				B	Mag. flux density	.....		.....	
10	$\sqrt{P}/L$	Electric force	H_e	Electric force	H_e	0	1/2	−3/2	H	Magnetic force	.....		$\sqrt{P}/L$	10
11	0	Resistance	R	.....		0	0	0	.....		.....		0	11
		Reactance	X	.....										
		Impedance	Z	.....										
		Conductance	G	.....										
		Susceptance	B	.....										
		Admittance	Y	.....										
12	L	Resistivity	ρ	.....		1	0	0	.....		.....		L	12
13	$1/L$	Conductivity	γ	.....		−1	0	0	.....		.....		$1/L$	13

(7) Capacitance C, Permeance P and Inductance L all have the dimensions (0, 0, 1) of time (seconds), The same is true of the well-known circuit time-constants CR and L/R.

(8) El. Quantity IT, Charge Q and static flux Φ_e together with Mag. Quantity m, Pole strength m, and Mag. Flux Φ have (1, 1/2, $-1/2$) or root time-energy $\sqrt{TW}$, or time root power $T\sqrt{P}$.

(9) El. and Mag. Surface densities and flux densities have (-1, 1/2, $-1/2$) or root time-energy per unit area, which conforms with $B^2/(2\mu_0)$ the well-known expression of El. or Mag. Energy per unit volume.

(10) The El. and Mag. Forces H_e and H, or gradients of potential, have (0, 1/2, $-3/2$) or $\sqrt{P}/L$. Their product $H_e H$ is thus P/S, which conforms with Poynting's Theorem.

(11) The Electrokinetic Resistances and Conductances have all zero dimensions, or are mere numerical coefficients of power delivery.

(12) El. Resistivity ρ has the dimensions of length.

(13) El. Conductivity γ has the dimensions of inverse length.

The question of the utility of the equidimensional convention may therefore be answered as follows:

(*a*) It simplifies and unifies the Table of Dimensions, which can then be readily memorized.

(*b*) Its dimensional expressions are mutually consistent and are closely associated with those of energy and power.

(*c*) It colligates and unites quantities in the electric and magnetic circuits of corresponding energy relations.

In comparing tables 1 and 2, it may be observed that the equidimensional series of table 2 gives, for each quantity, the arithmetical mean of the Maxwellian El. and Mag. dimensions in table 1. In other words, the equidimensional series is also the mean-dimension series.

The question as to what relations may exist between the equidimensional electromagnetic quantities and those in the actual physical universe, is much larger and more serious than any thus far here discussed. It is not easy to explain why the space constants elastivity and reluctivity should be velocities, capacitance and inductance times, and resistance only a coefficient. It may be that there remain yet other suppressed dimensions here ignored; or that the nature of electromagnetic phenomena may not admit of adequate representation in purely dynamic terms. For the present, we may be content to regard the equidimensional convention as a device for simplifying and condensing existing tables of electromagnetic dimensions.

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ON THERMODYNAMIC EQUILIBRIUM IN A STATIC EINSTEIN UNIVERSE

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§ 1. *Introduction.*—It has now become evident that the transformation of matter into radiation taking place throughout the universe and the red-shift observed in the light from the extra-galactic nebulae appear to imply a non-static quality in the universe which can be treated with some success with the help of a non-static cosmological line element.[1] If the quantity which gives the dependence of this non-static line element on the time is set equal to a constant, it is found that the line element then becomes the same as Einstein's original line element for a static universe. Hence the Einstein static universe may be regarded as a special case of the more general non-static universe, and we must continue to be interested in the properties of the Einstein universe not only because it is a limiting case of the more general model for the universe, but also because it represents a situation which might arise in the course of the evolution of the actual universe. The present article will deal with the thermodynamics of the Einstein universe, and in particular will treat the conditions for thermodynamic equilibrium between matter and radiation in such a universe assuming the possibility of their transformation into each other.

Treatments of the general problem of the equilibrium between matter and radiation have already been given for the case of a perfect monatomic gas interacting with black body radiation both in the absence and presence of gravitational fields. In the absence of any appreciable gravitational field, it was shown by the work of Stern[2] and myself[3] that the number of monatomic molecules of mass m present in unit volume at equilibrium at temperature T would be given by a formula of the form

$$N = bT^{3/2} e^{-\frac{mc^2}{kT}} \qquad (1)$$

where b is a constant whose value cannot be determined solely from the first and second laws of thermodynamics, c is the velocity of light and k is Boltzmann's constant. On account of the large effect of the exponent $-mc^2/kT$ the equilibrium concentration of matter given by this formula would be exceedingly low, even for masses as small as that of the electron and for temperatures as high as the 40,000,000° assumed in the interior of the stars, unless indeed the constant b could be shown to have an enormous value.

Also in the presence of the gravitational field of a spherical distribution of fluid, I have recently been able to show[4] that the equilibrium concentration of monatomic gas would again be given by a formula of the same form (1) as for flat space-time. But in the presence of the gravitational field of the Einstein static universe I originally found[5] a slightly different formula, which however still had a similar large exponential dependence on $-mc^2/kT$.

Since that time, however, the development of the non-static line element for the universe has made it easier to understand the process by which the Einstein universe could be regarded as changed from one static state to another and thus clarified the problem of determining a static state which would correspond to thermodynamic equilibrium. And the present article will show that a correct application of the principles of relativistic thermodynamics actually leads to exactly rather than merely approximately the same expression for the equilibrium concentration of gas in the Einstein static universe as formula (1) for the concentration in the absence of a gravitational field and for the concentration in a spherical mass of gravitating fluid.

§ 2. *The Einstein Line Element as a Special Case of the Non-Static Line Element.*—The line element for the non-static universe can be written in the form[6]

$$ds^2 = -\frac{e^{g(t)}}{[1 + r^2/4R^2]^2}(dx^2 + dy^2 + dz^2) + dt^2 \qquad (2)$$

where R is a constant, r is an abbreviation for $\sqrt{x^2 + y^2 + z^2}$ and the dependence of the line element on the time t is determined by the form of the function $g(t)$. This line element corresponds to a universe in which particles stationary with respect to x, y, z will remain stationary and corresponds to a distribution of matter and radiation of uniform proper density and pressure which, however, will in general be changing with the time if $g(t)$ is actually changing with the time.

If, however, $g(t)$ is given a constant value in the above expression, the line element reduces to that for the static Einstein universe, corresponding to a constant uniform density and pressure, and indeed can then easily be thrown into one of the more familiar forms for the Einstein line element

with the quantity $Re^{g/2}$ appearing as the radius of the universe. Thus for example if we take $g(t)$ as a constant g, and make the simple substitution

$$x = Xe^{-g/2}, \quad y = Ye^{-g/2}, \quad z = Ze^{-g/2} \tag{3}$$

we can write the line element in the form

$$ds^2 = -\frac{1}{\left[1 + \frac{X^2 + Y^2 + Z^2}{4(Re^{g/2})^2}\right]^2}(dX^2 + dY^2 + dZ^2) + dt^2 \tag{4}$$

which is one of the well-known expressions[7] for the Einstein line element with $Re^{g/2}$ as the radius.

Moreover, since the radius of the universe is the only adjustable parameter which occurs in the Einstein line element, it is now evident that we can regard a change in the Einstein universe from one static state to another as produced by a change in g from one given constant value to another. This makes it possible to apply to the change in state results which have been obtained from the study of non-static universes in which g is changing with the time.

§ 3. *Application of Relativistic Mechanics to Changes in State of the Einstein Static Universe.*—As shown in a previous article,[8] values for the energy-momentum tensor corresponding to the non-static line element (2) can be calculated from the principles of general relativity, and treating the material in the universe as a perfect fluid can be written in the form

$$8\pi T_1^1 = 8\pi T_2^2 = 8\pi T_3^3 = -8\pi p_0 = \frac{1}{R^2}e^{-g} + \ddot{g} + \frac{3}{4}\dot{g}^2 - \Lambda \tag{5}$$

$$8\pi T_4^4 = 8\pi \rho_{00} = \frac{3}{R^2}e^{-g} + \frac{3}{4}\dot{g}^2 - \Lambda \tag{6}$$

$$8\pi T_\rho^\sigma = 0 \ (\rho \neq \sigma) \tag{7}$$

where p_0 is the proper pressure and ρ_{00} the proper macroscopic density of the fluid, Λ is the cosmological constant and the dots indicate differentiation with respect to the time.

If we take g as a constant parameter in these equations, putting $\ddot{g} = \dot{g} = 0$, we obtain those relations between pressure, density and the metric which are necessary for a static Einstein universe.

On the other hand, if we let g vary with the time we can study the process through which the universe could change from one state to another, by substituting the values for the energy-momentum tensor given by equations (5), (6) and (7) into the well-known equation of relativistic mechanics

$$\frac{\partial \mathfrak{T}_\rho^\sigma}{\partial x_\sigma} - \frac{1}{2}\mathfrak{T}^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial x_\rho} = 0. \tag{8}$$

Doing so we merely obtain identities for the cases $\rho = 1, 2, 3$, but substituting for the case $\rho = 4$ we easily obtain the important result

$$\frac{d}{dt}\left(\rho_{00}e^{\frac{3\mu}{2}}\right) + p_0\frac{d}{dt}\left(e^{\frac{3\mu}{2}}\right) = 0, \tag{9}$$

where for simplicity we have set as an abbreviation

$$e^{\mu} = \frac{e^{g(t)}}{\left[1 + \frac{r^2}{4R^2}\right]^2}. \tag{10}$$

In accordance with the form of the line element, however, it is evident that $e^{\frac{3\mu}{2}}\,dx\,dy\,dz$ is the proper volume associated with the coördinate range $dx\,dy\,dz$. Hence the first term in equation (9) can be interpreted as the rate of increase in the proper energy in unit coördinate range and the second as the rate of expenditure of work on the surroundings. Furthermore, we remember that particles which are stationary with respect to x, y, z will remain stationary, so that the energy content of a given coordinate range will not be changing by the passage of matter through the boundaries of the coördinate range. Thus equation (9) has the simple physical interpretation that changes in the proper energy in any given coördinate range are due solely to the work done on the surroundings, and we can regard a change in the Einstein universe from one static state to another as the result of an *adiabatic* change in proper volume.

§ 4. *Application of the Relativistic Second Law of Thermodynamics to a Change in State of the Einstein Universe.* We must now consider the application of the second law of thermodynamics to changes in the state of an Einstein universe. In accordance with the principles of relativistic thermodynamics which I have previously developed,[9] the relativity analogue of the usual second law of thermodynamics can be expressed in the form

$$\frac{\partial}{\partial x_\rho}\left(\phi_0\sqrt{-g}\,\frac{dx_\rho}{ds}\right)dx_1\,dx_2\,dx_3\,dx_4 \geqq \frac{dQ_0}{T_0}, \tag{11}$$

where ϕ_0 is the proper density of entropy as measured by a local observer, dx_ρ/ds the macroscopic velocity of matter and T_0 the proper temperature, all at the point of interest, and dQ_0 is the heat measured in proper coordinates flowing into the infinitesimal region and in the time denoted by $dx_1dx_2dx_3dx_4$.

In applying this expression to the case of a change in state of the Einstein universe, we note that the macroscopic velocities dx_ρ/ds will be zero for the cases $\rho = 1, 2, 3$ since matter is stationary in the coördinate system x, y, z which is being used, and will be unity for the case $\rho = 4$ owing to

the form of the line element. We also note that dQ_0 will be zero, owing to the adiabatic character of the change pointed out in the last section. Hence the general principle reduces for our case to

$$\frac{d}{dt}\left(\phi_0\sqrt{-g}\right) = \frac{d}{dt}\left(\phi_0 e^{\frac{3\mu}{2}}\right) \geqq 0 \tag{12}$$

and since $\phi_0 e^{\frac{3\mu}{2}}$ is evidently the entropy associated with unit coördinate range, the result has the simple physical interpretation that the only possible changes in the state of an Einstein universe must be such as not to decrease the entropy content associated with each range of coördinates.

§ 5. *Conditions for Thermodynamic Equilibrium in a Static Einstein Universe.*—In accordance with the result of the foregoing section, the condition of thermodynamic equilibrium will evidently require that the quantity $\phi_0 e^{\frac{3\mu}{2}}$, which cannot decrease with the time, shall be a maximum at each point x, y, z. And noting equation (10) which defines μ, we can express this somewhat more conveniently by the requirement that $\phi_0 e^{\frac{3g}{2}}$ shall be a maximum.

Since we are interested, however, in the condition of thermodynamic equilibrium in a static Einstein universe, it is evident that the maximum value of this quantity must be achieved as a result of changes which preserve a static Einstein universe. In other words, when we vary the metrical quantity g and the pressure and density of matter and radiation in the universe in order to make $\phi_0 e^{\frac{3g}{2}}$ a maximum, we must preserve the truth of the equations of relativistic mechanics which relate the pressure and density to the metric in the way necessary to give a static Einstein universe, namely, as will be seen from § 3, the truth of equations (5) and (6) with g taken as independent of the time.

Hence we may now write as the desired conditions for thermodynamic equilibrium

$$\phi_0 e^{\frac{3g}{2}} \text{ to be a maximum} \tag{13}$$

under the subsidiary conditions obtained from (5) and (6)

$$p_0 + \frac{1}{8\pi R^2} e^{-g} - \frac{\Lambda}{8\pi} = 0 \tag{14}$$

and

$$\rho_{00} - \frac{3}{8\pi R^2} e^{-g} + \frac{\Lambda}{8\pi} = 0 \tag{15}$$

§ 6. *Equilibrium for a Monatomic Gas Interacting with Black Body*

Radiation.—We may now apply these conditions for thermodynamic equilibrium to the case of an Einstein universe filled with a monatomic gas interacting with black body radiation. Under these circumstances the proper entropy density ϕ_0, pressure p_0 and energy density ρ_{00} will evidently be functions of the proper temperature T_0 and number of atoms of gas N_0 per unit proper volume.

Hence employing the usual methods for treating a conditional maximum with the introduction of undetermined multipliers, we shall obtain from (13), (14) and (15) the following five equations

$$\begin{aligned}
e^{\frac{3g}{2}}\frac{\partial \phi_0}{\partial N_0} + \mu_1 \frac{\partial p_0}{\partial N_0} + \mu_2 \frac{\partial \rho_{00}}{\partial N_0} &= 0 \\
e^{\frac{3g}{2}}\frac{\partial \phi_0}{\partial T_0} + \mu_1 \frac{\partial p_0}{\partial T_0} + \mu_2 \frac{\partial \rho_{00}}{\partial T_0} &= 0 \\
\phi_0 \frac{\partial}{\partial g} e^{\frac{3g}{2}} + \frac{\mu_1}{8\pi R^2} \frac{\partial}{\partial g} e^{-g} - \frac{3\mu_2}{8\pi R^2} \frac{\partial}{\partial g} e^{-g} &= 0 \qquad (16) \\
p_0 + \frac{1}{8\pi R^2} e^{-g} - \frac{\Lambda}{8\pi} &= 0 \\
\rho_{00} - \frac{3}{8\pi R^2} e^{-g} + \frac{\Lambda}{8\pi} &= 0
\end{aligned}$$

for evaluating the undetermined multipliers μ_1 and μ_2 and the three unknown variables N_0, T_0 and g.

To treat these equations, we shall need explicit expressions for the quantities ϕ_0, p_0, and ρ_{00} in terms of the independent variables, N_0 the number of atoms of gas per unit volume and T_0 the proper temperature. For the proper density of entropy we can evidently write from the known expressions for the entropy of a monatomic gas and the entropy of radiation

$$\phi_0 = \frac{3}{2} N_0 k \log T_0 - N_0 k \log N_0 + N_0 k \log b e^{\frac{5}{2}} + \frac{4}{3} a T_0^3, \qquad (17)$$

where k is Boltzmann's constant, a is the Stefan-Boltzmann constant, and b is a constant of the right magnitude to assure the same starting point for the entropy of the gas and the radiation, the factor $e^{\frac{5}{2}}$ being introduced in the interests of simplicity of form in the final formula. Furthermore for the pressure p_0 and proper macroscopic density ρ_{00}, we evidently have the expressions

$$p_0 = N_0 k T_0 + \frac{1}{3} a T_0^4 \qquad (18)$$

and

$$\rho_{00} = N_0 m c^2 + \frac{3}{2} N_0 k T_0 + a T_0^4, \qquad (19)$$

where m is the mass of one atom and c is the velocity of light.

Introducing these equations into (17) and performing the indicated differentiations we then obtain the following five algebraic equations for the five unknown quantities N_0, T_0, g, μ_1 and μ_2.

$$
\begin{gathered}
e^{\frac{3g}{2}}\left(k\log\frac{bT_0^{\frac{3}{2}}}{N_0}+\frac{3}{2}k\right)+\mu_1 kT_0+\mu_2\left(mc^2+\frac{3}{2}kT_0\right)=0\\
e^{\frac{3g}{2}}\left(\frac{3}{2}\frac{N_0k}{T_0}+4aT_0^2\right)+\mu_1\left(N_0k+\frac{4}{3}aT_0^3\right)+\mu_2\left(\frac{3}{2}N_0k+4aT_0^3\right)=0\\
\frac{3}{2}e^{\frac{3g}{2}}\left(N_0k\log\frac{bT_0^{\frac{3}{2}}}{N_0}+\frac{5}{2}N_0k+\frac{4}{3}aT_0^3\right)-\frac{\mu_1}{8\pi R^2}e^{-g}+\frac{3\mu_2}{8\pi R^2}e^{-g}=0\\
N_0kT_0+\frac{a}{3}T_0^4+\frac{1}{8\pi R^2}e^{-g}-\frac{\Lambda}{8\pi}=0\\
N_0mc^2+\frac{3}{2}N_0kT_0+aT_0^4-\frac{3}{8\pi R^2}e^{-g}+\frac{\Lambda}{8\pi}=0.
\end{gathered}
\tag{20}
$$

These equations are rather complicated but by perfectly straightforward, although somewhat lengthy, algebraic manipulation they can be shown to be equivalent to the simpler set

$$
\begin{gathered}
N_0=bT_0^{\frac{3}{2}}e^{-\frac{mc^2}{kT_0}}\\
\mu_1=0\\
\mu_2=-\frac{e^{\frac{3g}{2}}}{T_0}\\
N_0kT_0+\frac{a}{3}T_0^4+\frac{1}{8\pi R^2}e^{-g}-\frac{\Lambda}{8\pi}=0;\\
N_0mc^2+\frac{3}{2}N_0kT_0+aT_0^4-\frac{3}{8\pi R^2}e^{-g}+\frac{\Lambda}{8\pi}=0,
\end{gathered}
\tag{21}
$$

five equations for the five unknown quantities N_0, T_0, g, μ_1 and μ_2 in terms of the constants a, b, k, m, Λ and R.

§ 7. *Conclusion.*—The first of these five equations gives the relation between equilibrium concentration N_0 of our monatomic gas and proper temperature T_0 which was the matter of chief interest for the present article. The relation has the same form as for the equilibrium concentration in flat space-time and in the gravitational field of a sphere of perfect fluid. We can again conclude that the equilibrium concentration of matter would be exceedingly low even for masses as small as that of the

electron and temperatures as high as 40,000,000° unless the constant b could be shown to have an enormous value.

[1] For references to the work of Friedman, LeMaitre, Robertson, Tolman, Eddington and de Sitter in this field, see Tolman, *Proc. Nat. Acad. Sci.*, **16**, 582 (1930).
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ANSWER TO PROF. STÖRMER'S REMARK

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Referring to my "Note on the Nature of Cosmic Rays,"[1] Prof. Störmer draws attention to the fact[2] that he had treated the problems of the motion of electrons in the magnetic field of the earth many years ago. He gives the complete list of his publications on the subject and, indeed, I must confess that I was not aware of his work on the particular phase of the problem to which my note is devoted.

But if I am guilty of having overlooked Prof. Störmer's priority, I may claim extenuating circumstances on several counts, and I believe that my note was not quite superfluous. *In the first place*, Prof. Störmer's papers appeared in magazines which are not readily accessible to the physicist. Even now, after the bibliography has been given by him, I have no access to that of his publications (Geneva, 1907) which contains the data, answering the questions put in my note, or the formulas from which these data could be derived. As all my colleagues in Southern California, and many in other places, are in the same position, it was well to restate the problem and its solution.

In the second place, Prof. Störmer's work dates from pre-relativistic days and is, therefore, based on classical mechanics while my note takes into account relativity. For the high velocities in question, one expects, at first sight, greatly different results. That Prof. Störmer's result is of interest also in the relativistic case is surprising and requires an explanation. The analysis can be based on the Hamilton-Jacobi partial differential equation. In the case of a magnetic dipole, acting upon one

electron, this differential equation has the same form for the classical and for the relativistic treatment. However, the two constants entering into the equation are widely different in the two treatments. The limiting value of the distance from the magnetic pole, which was determined by Prof. Störmer and by myself, depends only on one of these constants denoted by A in my note. The physical meaning of this constant is the absolute value of the momentum. In terms of the kinetic energy α and of the rest mass m, it has the following expressions

$$A^2 = 2m\alpha + \alpha^2/c^2, \quad \text{relativistic}$$
$$A^2 = 2m\alpha, \quad \text{classical}$$

If we express the limiting angle as a function of the energy or of the velocity of the electrons, the results in the two theories are vastly different. However, Prof. Störmer expresses it as a function of the "magnetic rigidity" aH, where a means the radius of the circle which the electron would describe in a homogeneous field of the strength H. The connection between the magnetic rigidity and the momentum is given by the equations (notations as in the note)

$$mv/\sqrt{1-\beta^2} = eaH/c, \quad \text{relativistic}$$
$$mv = eaH/c, \quad \text{classical}$$

The left sides of both equations represent the momentum and are equal to our constant A. We find, therefore, in both cases

$$A = aHe/c.$$

The constant A is, therefore, directly proportional to the magnetic rigidity. If we plot the maximum angle against the magnetic rigidity we get the same curve in the classical and in the relativistic treatment.

In conclusion, I take the opportunity for correcting an erratum: in formulas (18) and (21) of the "Note," read under the square root $1 + \sin^3\vartheta$ instead of $1 + \sin^2\vartheta$.

[1] P. S. Epstein, these PROCEEDINGS, **16**, p. 658, 1930.

[2] Carl Störmer, *Ibid.*, **17**, p. 62, 1931.